Lecture 14: Pseudo-random Generators

Definition (PRG)

Let $G \colon \{0,1\}^n \to \{0,1\}^{n+\ell}$ be a function that is efficient to evaluate. We say that G is a pseudorandom generator, if

1. The stretch $\ell > 0$, and
2. The distribution $G(\mathbb{U}^n_{\{0,1\}})$ "appears indistinguishable" from the distribution $\mathbb{U}_{\{0,1\}^{n+\ell}}$ for computationally bounded adversaries.

Clarifications.

1. The input bits $s \sim \mathbb{U}_{\{0,1\}^n}$ that is fed to the PRG is referred to as the seed of the PRG
2. Intuition of a PRG: We rely on a small amount of pure randomness to jumpstart a PRG that yields more (appears to be) random bits

3. Note that if $\ell \leqslant 0$ then PRG is easy to construct. Note that in this case $n + \ell \leqslant n$. So, $G(s)$ just outputs the first $n + \ell$ bits of the input seed s.
4. The entire non-triviality is to construct G when $\ell \geqslant 1$. Suppose $\ell = 1$. Note that in the case G has 2^n different possible inputs. So, G has at most 2^n different possible outputs. The range $\{0,1\}^{n+\ell}$ has size 2^{n+1}. So, there are at least $2^{n+1} - 2^n = 2^n$ elements in the range that have no pre-image under the mapping G. We can conclude that $G(\mathbb{U}_{\{0,1\}^n})$ assigns 0 probability to at least 2^n entries in the range.
5. Note that the distribution $G(\mathbb{U}_{\{0,1\}^n})$ is different from the distribution $\mathbb{U}_{\{0,1\}^{n+1}}$. A computationally unbounded adversary can distinguish $G(\mathbb{U}_{\{0,1\}^n})$ from $\mathbb{U}_{\{0,1\}^{n+1}}$. However, for a computationally bounded adversary, the distribution $G(\mathbb{U}_{\{0,1\}^n})$ appears same as the distribution $\mathbb{U}_{\{0,1\}^{n+1}}$

6. In this class, we shall see a construction of PRG when $\ell = 1$ given a OWP f. In general, we know how to construct a PRG using a OWF. However, presenting that construction is beyond the scope of this course.
7. Note that these PRG constructions work for ny OWF f. So, if some OWF f is broken in the future due to progress in mathematics or use of quantum computers, then we can simply replace the existing PRG constructions to use a different OWF g.

Observation on Bijections

- Let $f\colon \{0,1\}^n \to \{0,1\}^n$ be a bijection
- Suppose we sample $x \xleftarrow{\$} \{0,1\}^n$
- For any $y \in \{0,1\}^n$, what is the probability that $f(x) = y$?
  - Note that there is a unique x' such that $f(x') = y$, because f is a bijection
  - $f(x) = y$ if and only if $x = x'$, i.e. the probability that $f(x) = y$ is $1/2^n$.
- So, the distribution of $f(x)$, where $x \xleftarrow{\$} \{0,1\}^n$, is a uniform distribution over $\{0,1\}^n$

Goldreich-Levin Hardcore Predicate I

- We define the inner product of $r \in \{0,1\}^n$ and $x \in \{0,1\}^n$ as $\langle r, x\rangle = r_1x_1 \oplus r_2x_2 \oplus \cdots \oplus r_nx_n$
- We will state the Goldreich-Levin Hardcore Predicate without proof

Theorem (Goldrecih-Levin Hardcore Predicate)

If $f\{0,1\}^n \to \{0,1\}^n$ is a one-way function then the bit $b = \langle r, x\rangle$ cannot be predicted given $(r, f(x))$. This proof is beyond the scope of this course. However, students are encouraged to study this celebrated result in the future.

Goldreich-Levin Hardcore Predicate II

A note on "Predicting a bit"

- Note that it is trivial to correctly predict any bit with probability $1/2$. (Guess a uniformly random bit z. The probability that z is identical to the hidden bit is $1/2$)
- To non-trivially predict a hidden bit, the adversary has to correctly predict it with probability at least $1/2 + \varepsilon$, where $\varepsilon = 1/\mathrm{poly}(n)$

One-bit Extension PRG I

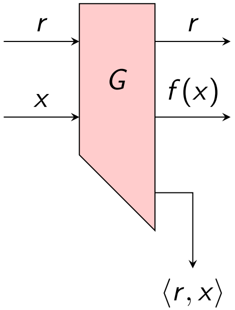
- Recall: A pseudorandom generator (PRG) is a function $G_{n,n+\ell}\colon \{0,1\}^n \to \{0,1\}^{n+\ell}$ such that, for $x \xleftarrow{\$} \{0,1\}^n$, the output $G_{n,n+\ell}(x)$ *looks like* a random $(n+\ell)$-bit string.
- A one-bit extension PRG has $\ell = 1$
- Suppose $f\colon \{0,1\}^n \to \{0,1\}^n$ is a OWP (i.e., f is a OWF and it is a bijection)
- Note that the mapping $(r,x) \mapsto (r, f(x))$ is a bijection
- So, the output $(r, f(x))$ is a uniform distribution if $(r,x) \xleftarrow{\$} \{0,1\}^{2n}$
- Now, the output $(r, f(x), \langle r, x\rangle)$ looks like a random $(2n+1)$-bit string if f is a OWP (because of Goldreich-Levin Hardcore Predicate result)

One-bit Extension PRG II

- Consider the function $G_{2n,2n+1}\colon \{0,1\}^{2n} \to \{0,1\}^{2n+1}$ defined as follows

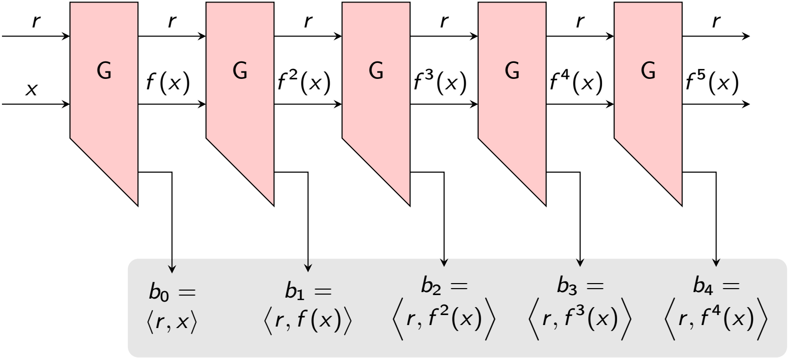
$$G_{2n,2n+1}(r,x) = (r, f(x), \langle r,x \rangle)$$

- This is a one-bit extension PRG if f is a OWP
- This construction will be pictorially represented as follows



Generating Long Pseudorandom Bit-Strings I

- In the previous step, we saw how to construct a one-bit extension PRG G
- Now, we use the previous step iteratively to construct arbitrarily long pseudorandom bit-strings
- The next slide, using the one-bit extension PRG, provides the intuition to construct $G_{2n,\ell} \colon \{0,1\}^{2n} \to \{0,1\}^{2n+\ell}$, for arbitrary $\ell = \mathrm{poly}(n)$.
- The example shows only $\ell = 5$ but can be extended naturally to arbitrary $\ell = \mathrm{poly}(n)$


r
x
G
r
f(x)
G
r
f²(x)
G
r
f³(x)
G
r
f⁴(x)
G
r
f⁵(x)
b₀ = ⟨r, x⟩
b₁ = ⟨r, f(x)⟩
b₂ = ⟨r, f²(x)⟩
b₃ = ⟨r, f³(x)⟩
b₄ = ⟨r, f⁴(x)⟩

Length Doubling PRG

- This is a PRG that takes n-bit seed and outputs $2n$-bit string
- $G_{n,2n}$ is a length-doubling PRG if $G_{n,2n} \colon \{0,1\}^n \to \{0,1\}^{2n}$ and $G_{n,2n}$ is a PRG
- We can use the iterated construction in the previous slide to construct a length-doubling PRG from one-bit extension PRG

Food for thought

- Design secret-key encryption schemes where the message is much longer than the secret key